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AN ERROR ESTIMATE OF THE LEAST SQUARES FINITE ELEMENT METHOD FOR THE STOKES PROBLEM IN THREE DIMENSIONS

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ABSTRACT. In this paper we are concerned with the Stokes problem in three dimensions (see recent works of the author and B. N. Jiang for the two-dimensional case). It is a linear system of four PDEs with velocity $\underline{u}$ and pressure p as unknowns. With the additional variable $\underline{\omega} = \operatorname{curl} \underline{u}$, the second-order problem is reduced to a first-order system. Considering the compatibility condition $\operatorname{div} \underline{\omega} = 0$, we have a system with eight first-order equations and seven unknowns. A least squares method is applied to this extended system, and also to the corresponding boundary conditions. The analysis based on works of Agmon, Douglis, and Nirenberg; Wendland; Zienkiewicz, Owen, and Niles; etc. shows that this method is stable in the h-version. For instance, if we choose continuous piecewise polynomials to approximate $\underline{u}$, $\underline{\omega}$, and p, this method achieves optimal rates of convergence in the H^1-norms.

INTRODUCTION

Let Ω be an open bounded and connected subset of $\mathbb{R}^3$ with a smooth boundary Γ. Let $\underline{f} \in [L^2(\Omega)]^3$ be a given function representing the body force. The Stokes problem can be posed as

$$(1.1) \qquad \begin{cases} -\nu \Delta \underline{u} + \operatorname{grad} p = \underline{f} & \text{in } \Omega, \\ \operatorname{div} \underline{u} = 0 & \text{in } \Omega, \\ \underline{u} = 0 & \text{on } \Gamma, \end{cases}$$

where $\underline{u}, p$ with $(p, 1) = 0$, and ν are respectively velocity, pressure, and kinematic viscosity (constant), all of which are assumed to be nondimensionalized.

Over the past two decades many engineers and mathematicians have studied the above problem. The mixed Galerkin method solves this problem successfully. In most cases the elements are required to satisfy a saddle point condition [4, 5, 8, 9, 22], which is not necessary for our method.

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Introducing $\underline{\omega} = \operatorname{curl} \underline{u}$, we can transform (1.1) into the first-order system

$$(1.2) \qquad L\underline{U} = \begin{bmatrix} \nu \operatorname{cur} \underline{\omega} + \operatorname{grad} p \\ -\operatorname{div} \underline{\omega} \\ \nu \operatorname{curl} \underline{u} - \nu \underline{\omega} \\ -\operatorname{div} \underline{u} \end{bmatrix} = \begin{bmatrix} \underline{f} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{in } \Omega$$

and the boundary condition

$$R\underline{U} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & n_1 & n_2 & n_3 & 0 \end{bmatrix} \underline{U} = \underline{0} \quad \text{on } \Gamma,$$

where $\underline{U} = [\underline{u}, \underline{\omega}, p]^t$. The above system has been weighted, which is required by the analysis in the following sections. The relation $\operatorname{div} \underline{\omega} = 0$ is the compatibility condition; without it, the numerical scheme may not be convergent.

The boundary condition $\underline{u} = \underline{0}$ on Γ implies that the tangential derivatives of u_i vanish, or $\nabla u_i \times \underline{n} = \underline{0}$ for $i = 1, 2, 3$ and

$$\begin{aligned} \underline{\omega} \cdot \underline{u} &= \left(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}\right) n_1 + \left(\frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x}\right) n_2 + \left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right) n_3 \\ &= \left(\frac{\partial u_3}{\partial y} n_1 - \frac{\partial u_3}{\partial x} n_2\right) + \left(\frac{\partial u_2}{\partial x} n_3 - \frac{\partial u_2}{\partial z} n_1\right) + \left(\frac{\partial u_1}{\partial z} n_2 - \frac{\partial u_1}{\partial y} n_3\right) \\ &= 0. \end{aligned}$$

The least squares method relaxes the boundary conditions and the exact divergence-free condition, so that the elements require less restriction. For example, if all of the $\underline{u}$, $\underline{\omega}$, and p are allowed to be approximated by piecewise linear functions in $H^1(\Omega)$, we will show that the method achieves an optimal rate of convergence.

Weighted least squares methods were used by Bramble, Nitsche, Schatz, Fix, Gunzburger, Nicolaides, Oden, Carey, Zienkiewicz, and many others [7, 15] in [2, 12]. In this paper we are going to apply the theory of Agmon-Douglis-Nirenberg-type first-order linear systems to the weighted least squares methods. The work of Aziz, Kellogg, Stephens, and Wendland [2, 23] gave a general theory for this method. Jiang, Povinelli, and Chang [18–19] have successfully transformed the Stokes problem into a first-order system in a two-dimensional region and then treated it by a least squares method.

In this paper we will present not only the numerical least squares scheme, but also derive error estimates in the three-dimensional case.

2. Notation and formulation of the problem

Throughout this paper, we will employ standard notation for Sobolev spaces and their associated norms [14, 22]. We let $H^m(\Omega)$ denote the Sobolev space of functions having square integrable derivatives of order up to m over Ω,

$$(2.1) \qquad H^m(\Omega) = \{v \in L^2(\Omega);\ \partial^\alpha v \in L^2(\Omega) \text{ for } |\alpha| \le m\},$$

where $\alpha = (\alpha_1\alpha_2\alpha_3)$, $\partial^\alpha v = \partial^{|\alpha|}v/\partial x^{\alpha_1}\partial y^{\alpha_2}\partial z^{\alpha_3}$, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$. We define the inner product and the norm in $H^m(\Omega)$ as

$$(2.2) \qquad (u, v)_m = \sum_{|\alpha| \le m} \int_\Omega \partial^\alpha u \cdot \partial^\alpha v$$

and

$$\|u\|_m^2 = (u, u)_m.$$

The space $H_0^m(\Omega)$ is the closure of $\mathscr{D}(\Omega)$ for the norm $\|\cdot\|_m$, where $\mathscr{D}(\Omega)$ is the linear space of functions infinitely differentiable and with compact support on Ω. We denote by $H^{-m}(\Omega)$ the dual space of $H_0^m(\Omega)$ normed by

$$(2.3) \qquad \|u\|_{-m} = \sup \frac{|(u, v)|}{\|v\|} \quad \text{over } v \in H_0^m(\Omega) \text{ and } v \not\equiv 0.$$

The trace operator $\gamma_0\colon H^1(\Omega) \to L^2(\Gamma)$ is a bounded linear operator agreeing with the restriction operator $u \mapsto u/\Gamma$ for continuous functions on $\overline{\Omega}$. The kernel of γ_0 is $H_0^1(\Omega)$, and the image is denoted by $H^{1/2}(\Gamma)$, which is also a Hilbert space; we define its norm by

$$(2.4) \qquad \|u\|_{1/2,\Gamma} = \{\inf \|v\|_1 ;\ v \in H^1(\Omega) \text{ and } \gamma_0 U = u \text{ on } \Gamma\}.$$

The trace inequality shows that there exists $c > 0$ independent of v such that

$$(2.5) \qquad \|\gamma_0 v\|_{1/2,\Gamma} \le c\|v\|_1 \quad \text{for any } v \in H^1(\Omega).$$

We define the function space

$$(2.6) \qquad V = \{\underline{v} \in [H^1(\Omega)]^7\}.$$

Following the work of Bramble and Scott [6], we will use a finite-dimensional subspace $V_r^h \in V$ of functions to approximate our solutions. The parameter h, which represents a mesh spacing, is used to indicate the approximation property of V_r^h. We say that V_r^h approximates optimally with respect to r if for every $\underline{v} \in V \cap [H^{r+1}(\Omega)]^7$ there exists $\underline{v}^h \in V_r^h$ such that

$$(2.7) \qquad h\|\underline{v} - \underline{v}^h\|_1 + \|\underline{v} - \underline{v}^h\|_0 \le Ch^{r+1}\|\underline{v}\|_{r+1},$$

where the positive constant C is independent of $\underline{v}$ and h.

We then define the least squares quadratic functional

$$(2.8) \qquad J(\underline{v}) = \int_\Omega \left[L\underline{v} - \begin{pmatrix} f \\ \underline{0} \end{pmatrix}\right] \cdot \left[L\underline{v} - \begin{pmatrix} f \\ \underline{0} \end{pmatrix}\right] + h^{-1} \int_\Gamma R\underline{v} \cdot R\underline{v} \quad \text{for } \underline{v} \in V.$$

If $\underline{U}$ minimizes $J(\underline{v})$ over $\underline{v} \in V$, it is easy to see that

$$(2.9) \qquad \int_\Omega L\underline{U} \cdot L\underline{v} + h^{-1} \int_\Gamma R\underline{U} \cdot R\underline{v} = \int_\Omega \begin{pmatrix} f \\ \underline{0} \end{pmatrix} \cdot L\underline{v} \quad \text{for any } \underline{v} \in V,$$

so a solution of (1.2) is also a solution of (2.9), and a sufficiently smooth solution of (2.9) is also a solution of (1.2).

A finite element approximation to the solution of (1.2) or (2.9) is defined as a solution of the problem

$$(2.10) \qquad \operatorname{Min} J(\underline{v}^h) \quad \text{over } \underline{v}^h \in V_r^h.$$

Similarly to (2.9), the solution $\underline{U}^h$ of (2.10) satisfies the corresponding finite algebraic equations

$$(2.11)\qquad \int_\Omega L\underline{U}^h \cdot L\underline{v}^h + h^{-1}\int_\Gamma R\underline{U}^h \cdot R\underline{v}^h = \int_\Omega \begin{pmatrix} \underline{f} \\ \underline{0} \end{pmatrix} \cdot L\underline{v}^h \quad \text{for any } \underline{v}^h \in V_r^h.$$

Once a basis for V_r^h is chosen, (2.11) becomes a symmetric linear algebraic system. Moreover, in §4 we will show that this algebraic system is also positive definite.

3. The a priori estimates

In this section we consider an auxiliary elliptic system obtained from the system (1.2) by adding another unknown. The a priori inequality associated with this "enriched" system is important in the derivation of our error estimate. The "enriched" set of unknowns is $\underline{\mathscr{U}} = [\underline{u}, k, \underline{\omega}, p]^t$, where k is a new variable that plays a role similar to that of a slack variable in linear programming. The enriched differential system is defined in terms of the operator L_e by

$$(3.1)\qquad L_e\underline{\mathscr{U}} = \begin{bmatrix} \nu \operatorname{curl} \underline{\omega} + \operatorname{grad} p \\ -\operatorname{div} \underline{\omega} \\ \nu \operatorname{curl} \underline{u} + \operatorname{grad} k - \nu \underline{\omega} \\ -\operatorname{div} \underline{u} \end{bmatrix} = \underline{F} \quad \text{in } \Omega,$$

while the enriched boundary conditions are given in terms of the boundary operator R_e by

$$(3.2)\qquad \begin{aligned} R_e\underline{\mathscr{U}} &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & n_1 & n_2 & n_3 & 0 \end{bmatrix} \underline{\mathscr{U}} \\ &= \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ n_1\omega_1 + n_2\omega_2 + n_3\omega_3 \end{bmatrix} = \underline{0} \quad \text{on } \Gamma. \end{aligned}$$

If $\int_\Omega p = \int_\Omega k = 0$ and the compatibility conditions on the data, $\int_\Omega f_4 = \int_\Omega f_8 = 0$, are satisfied, the boundary value problem defined by (3.1), (3.2) is well posed, and the a priori inequality associated with this problem gives rise to our desired inequality for the Stokes problem. Taking the div of both sides of equations 5 through 7 in (3.1) yields $\Delta k = 0$; evaluating the normal component of the vector consisting of the same three rows of $L_e\underline{\mathscr{U}}$ anywhere on the boundary yields $\frac{\partial k}{\partial n} = 0$,[1] and together with $\int_\Omega k = 0$ this finally gives $k \equiv 0$. Equation (3.1) can thus be rewritten as

$$(3.3)\qquad L_e\underline{\mathscr{U}} = A\underline{\mathscr{U}}_x + B\underline{\mathscr{U}}_y + C\underline{\mathscr{U}}_z + D\underline{\mathscr{U}} = \underline{F},$$

where A, B, C, and D are 8×8 constant matrices:

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$$

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[1] $\nabla k \cdot \underline{n} = \nu(\underline{\omega} - \operatorname{curl} \underline{u}) \cdot \underline{n} = \nu(\underline{\omega} \cdot \underline{n} - \sum_{l=1}^3 [\nabla u_i \times \underline{n}]_l) = \nu(0 - 0) = 0$, since $\underline{\omega} \cdot \underline{n} = 0$ and $\underline{u} = \underline{0}$ on Γ.

and the same for B, C, and D, with

$$A_2 = A_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -\nu & 0 \\ 0 & \nu & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \qquad A_1 = A_4 = [0]_{4\times 4},$$

$$B_2 = B_3 = \begin{bmatrix} 0 & 0 & \nu & 0 \\ 0 & 0 & 0 & 1 \\ -\nu & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \qquad B_1 = B_4 = [0]_{4\times 4},$$

$$C_2 = C_3 = \begin{bmatrix} 0 & -\nu & 0 & 0 \\ \nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \qquad C_1 = C_4 = [0]_{4\times 4},$$

and

$$D_4 = \begin{bmatrix} -\nu & 0 & 0 & 0 \\ 0 & -\nu & 0 & 0 \\ 0 & 0 & -\nu & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad D_1 = D_2 = D_3 = [0]_{4\times 4}.$$

The operator L_e in (3.1) has the matrix form

$$L_e = [l_{ij}(\partial)] = A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y} + C\frac{\partial}{\partial z} + D.$$

We also define the space for the enriched variables as

(3.4) $$V_e = \{\underline{v} \in [H^1(\Omega)]^8\}.$$

Following the procedure in [1], we check the polynomials $l_{ij}(\underline{\Theta})$. We find that there exist integer weights $s_i = 0$, $t_j = 1$, $i, j = 1, 2, \dots, 8$, such that

$$\deg[l_{ij}(\underline{\Theta})] \le s_i + t_j \quad \text{for } i, j = 1, 2, \dots, 8,$$

where $\underline{\Theta} = [x\,y\,z]^t$ is the spacial variable. We define l'_{ij} to be the polynomial with the terms in l_{ij} which are precisely of the order $s_i + t_j$,

$$\begin{aligned} \mathscr{L}(\underline{\Theta}) \equiv \det[l'_{ij}(\underline{\Theta})] &= \det(xA + yB + zC) = \nu^4(x^2+y^2+z^2)^4 \\ &\neq 0 \quad \text{for real } \underline{\Theta} = [x\,y\,z]^t \neq \underline{0}. \end{aligned}$$

By the theory of Agmon, Douglis, and Nirenberg [1], the operator L_e defined in (3.1) is an elliptic system and is also uniformly elliptic by the definition given in [20]. In this paper, since we discuss the problem with constant coefficients, the position variable P is dropped. We state the supplementary condition, which is fulfilled for our problem, since we have three independent variables.

Supplementary Condition. $\mathscr{L}(\underline{\Theta})$ is of even degree $2m$. For any pair of linearly independent real vectors $\underline{\Theta}$ and $\underline{\Theta}'$, the polynomial $\mathscr{L}(\underline{\Theta}+\tau\underline{\Theta}')$ in the complex variable τ has exactly m roots with positive imaginary part.

Next, we check the boundary condition to see whether it satisfies the complementing condition.

The operator R_e in (3.2) involves a constant matrix of order 4×8. The order of the boundary operator R_e depends on two systems of integer weights, in this case the system t_j, $j = 1, 2, \ldots, 8$, already attached to the dependent variables, and a new system r_h, $h = 1, 2, 3, 4$, of which r_h pertains to the hth condition in (3.2). In this paper we simply take $r_h = -1$, $h = 1, \ldots, 4$. Let $R'_{hj}(\underline{\Theta})$ consist of the terms in $R_{hj}(\underline{\Theta})$ which are precisely of the order $r_h + t_j$. There is no difference between R_{hj} and R'_{hj} in our problem.

At any point P on a regular portion of Γ, let $\underline{n}$ denote the outer normal at P, and $\underline{\Theta} \neq 0$ any tangent to Γ. Denote by $\tau_h^+(\underline{\Theta})$, $h = 1, 2, 3, 4$, the four roots in τ with positive imaginary part of the characteristic equation $\mathscr{L}(\underline{\Theta} + \tau\underline{n}) = 0$. The existence of these roots is assured by the Supplementary Condition. Set

$$M^+(P, \underline{\Theta}, \tau) = \prod_{h=1}^{4} (\tau - \tau_h^+(P, \underline{\Theta})). \tag{3.5}$$

In our case, $\mathscr{L}(\underline{\Theta} + \tau\underline{n}) = 0$ implies

$$((\theta_1 + \tau n_1)^2 + (\theta_2 + \tau n_2)^2 + (\theta_3 + \tau n_3)^2)^4 = 0,$$

or $(\tau^2 + 1)^4 = 0$, so $M^+(P, \underline{\Theta}, \tau) = (\tau - i)^4$. Let $(L^{jk}(\underline{\Theta}))$ denote the matrix adjoint to $(l'_{ij}(\underline{\Theta}))$,

$$(l'_{ij}(\underline{\Theta})) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -\nu z & \nu y & x \\ 0 & 0 & 0 & 0 & \nu z & 0 & -\nu x & y \\ 0 & 0 & 0 & 0 & -\nu y & \nu x & 0 & z \\ 0 & 0 & 0 & 0 & -x & -y & -z & 0 \\ 0 & -\nu z & \nu y & x & 0 & 0 & 0 & 0 \\ \nu z & 0 & -\nu x & y & 0 & 0 & 0 & 0 \\ -\nu y & \nu x & 0 & z & 0 & 0 & 0 & 0 \\ -x & -y & -z & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

After tedious elementary operations, the adjoint matrix to $(l'_{ij}(\underline{\Theta}))$ is seen to be

$$(L^{jk}(\underline{\Theta})) = \nu^3(x^2+y^2+z^2)^3 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -z & y & \nu x \\ 0 & 0 & 0 & 0 & z & 0 & -x & \nu y \\ 0 & 0 & 0 & 0 & -y & x & 0 & \nu z \\ 0 & 0 & 0 & 0 & -\nu x & -\nu y & -\nu z & 0 \\ 0 & -z & y & \nu x & 0 & 0 & 0 & 0 \\ z & 0 & -x & \nu y & 0 & 0 & 0 & 0 \\ -y & x & 0 & \nu z & 0 & 0 & 0 & 0 \\ -\nu x & -\nu y & -\nu z & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3.6}$$

The above-mentioned criterion for the boundary problem (3.1), (3.2) to be coercive is that the following algebraic condition be satisfied.

Complementing Boundary Condition. For any $P \in \Gamma$ and any real, nonzero vector $\underline{\Theta}$ tangent to Γ at P, regard $M^+(P, \underline{\Theta}, \tau)$ and the elements of the matrix

$$\sum_{j=1}^{N} R'_{hj}(P, \underline{\Theta} + \tau\underline{n}) L^{jk}(P, \underline{\Theta} + \tau\underline{n}) \tag{3.7}$$

as polynomials in the indeterminate τ. The rows of the latter matrix are required to be linearly independent modulo $M^+(P, \underline{\Theta}, \tau)$, i.e.,

$$\sum_{h=1}^{m} C_h R'_{hj} L^{jk} \equiv 0 \quad (\operatorname{Mod} M^+) \tag{3.8}$$

only if the constants C_h are all zero.

Without loss of generality, let the tangent vector $\underline{\theta} = (\theta_1, \vartheta_2, \theta_3)$ and the normal vector $\underline{n} = (n_1, n_2, n_3)$ be unit vectors. In our case, $N = 8$. From (3.2) and (3.6) we find that the matrix defined in (3.7) for our problem is

$$\nu^3(\tau^2+1)^3 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -m_3 & m_2 & \nu m_1 \\ 0 & 0 & 0 & 0 & m_3 & 0 & -m_1 & \nu m_2 \\ 0 & 0 & 0 & 0 & -m_2 & m_1 & 0 & \nu m_3 \\ \theta_3 n_2 - \theta_2 n_3 & \theta_1 n_3 - \theta_3 n_1 & \theta_2 n_1 - \theta_1 n_2 & \nu\tau & 0 & 0 & 0 & 0 \end{bmatrix},$$

where

$$\begin{cases} m_1 = \theta_1 + \tau n_1, \\ m_2 = \theta_2 + \tau n_2, \\ m_3 = \theta_3 + \tau n_3. \end{cases}$$

Assume the condition (3.8) is fulfilled. In terms of the matrix entries, the condition implies that there are constants $D_1, D_2, \ldots, D_8$ such that

$$\begin{aligned} C_4\nu^3(\tau^2+1)^3(\theta_3 n_2 - \theta_2 n_3) &= D_1 M_+(\tau), \\ C_4\nu^3(\tau^2+1)^3(\theta_1 n_3 - \theta_3 n_1) &= D_2 M_+(\tau), \\ C_4\nu^3(\tau^2+1)^3(\theta_2 n_1 - \theta_1 n_2) &= D_3 M_+(\tau), \\ C_4\nu^4(\tau^2+1)^3\tau &= D_4 M_+(\tau), \\ \nu^3(\tau^2+1)^3(C_2 m_3 - C_3 m_2) &= D_5 M_+(\tau), \\ \nu^3(\tau^2+1)^3(C_3 m_1 - C_1 m_3) &= D_6 M_+(\tau), \\ \nu^3(\tau^2+1)^3(C_1 m_2 - C_2 m_1) &= D_7 M_+(\tau), \\ \nu^4(\tau^2+1)^3(C_1 m_1 + C_2 m_2 + C_3 m_3) &= D_8 M_+(\tau). \end{aligned}$$

The roots of the polynomial $M_+(\tau)$ have positive imaginary parts with multiplicity 4. On the left-hand sides we have the factor of positive imaginary parts with multiplicity 3 only, hence $D_1, D_2, \ldots, D_8 = 0$, and $C_4 = 0$. From the last four equations above we have

$$\begin{aligned} (C_1, C_2, C_3)^T \times (m_1, m_2, m_3)^T &= \underline{0}, \\ (C_1, C_2, C_3)^T \cdot (m_1, m_2, m_3)^T &= 0. \end{aligned}$$

Since $(m_1, m_2, m_3)^T \not\equiv \underline{0}$, we have $C_1 = C_2 = C_3 = 0$, and the Complementing Boundary Condition is indeed satisfied.

By the work in [1], we can now state

Theorem 1. *For $l \geq 0$ there is a constant $C > 0$ such that*

$$\|\mathscr{U}\|_{l+1} \leq C(\|L_e\mathscr{U}\|_l + \|R_e\mathscr{U}\|_{l+\frac{1}{2}} + \|\mathscr{U}\|_0). \tag{3.9}$$

It can be shown that the boundary value problem associated with (3.1), (3.2) has a unique solution. Therefore, the term $\|\mathscr{U}\|_0$ can be dropped from (3.9). If the resulting inequality is applied with $k = 0$, we obtain the a priori inequality

$$\|\mathscr{U}\|_{l+1} \leq C(\|L_e\mathscr{U}\|_l + \|R_e\mathscr{U}\|_{l+\frac{1}{2}}). \tag{3.10}$$

The inequality (3.10) is crucial for our least squares error analysis. It is interesting to note that (3.10) contains, in particular, the usual shift inequality for the system (1.1). Let $[\underline{u}, p]$ solve (1.1), and let $\underline{\omega} = \operatorname{curl} \underline{u}$. Then $\underline{\mathscr{U}} = [\underline{u}, k = 0, \underline{\omega}, p]^t$ satisfies $L_e\underline{\mathscr{U}} = [\underline{f}, 0, \underline{0}, 0]^t$ and $R_e\underline{\mathscr{U}} = [\underline{0}, 0]^t$. Hence (3.10) yields

$$\|\underline{u}\|_{l+1} + \|\operatorname{curl} \underline{u}\|_{l+1} + \|p\|_{l+1} \leq C\|\underline{f}\|_l. \tag{3.11}$$

Since $\operatorname{div} \underline{u} = 0$, (3.11) yields the usual a priori inequality,

$$\|\underline{u}\|_{l+2} + \|p\|_{l+1} \leq C\|\underline{f}\|_l,$$

for solutions of (1.1).

4. Error estimates

In this section we will discuss the numerical scheme defined by (2.11). Denote the bilinear form

$$a(\underline{U}, \underline{V}) = \int_\Omega L\underline{U} \cdot L\underline{V} + h^{-1} \int_\Gamma R\underline{U} \cdot R\underline{V}. \tag{4.1}$$

Thus, (2.9) and (2.11) can be reformulated as follows: find $\underline{U} \in V$ (defined by (2.6)) such that

$$a(\underline{U}, \underline{V}) = \int_\Omega \begin{pmatrix} \underline{f} \\ \underline{0} \end{pmatrix} \cdot L\underline{V} \quad \text{for any } \underline{V} \in V, \tag{4.2}$$

and find $\underline{U}^h \in V_r^h$ (defined by (2.7)), such that

$$a(\underline{U}^h, \underline{V}^h) = \int_\Omega \begin{pmatrix} \underline{f} \\ \underline{0} \end{pmatrix} \cdot L\underline{V}^h \quad \text{for any } \underline{V}^h \in V_r^h. \tag{4.3}$$

By inspection, a is symmetric and $a(\underline{U}, \underline{U}) \geq 0$. Furthermore, if $a(\underline{U}, \underline{U}) = 0$, from (3.10) we get $\underline{U} = 0$. Hence, the matrix associated with the linear system (2.11) is positive definite.

Combining (4.2), (4.3), we have

$$a(\underline{U} - \underline{U}^h, \underline{V}^h) = 0 \quad \text{for any } \underline{V}^h \in V_r^h. \tag{4.4}$$

To obtain an error estimate for our least squares method, we shall require an "inverse assumption" on the subspace V_r^h. Inverse assumptions are common in least squares analyses; see, for example, [2, 7]. The property we need is the existence of a constant $C > 0$ such that

$$\|R\underline{V}^h\|_{1/2,\Gamma} \leq Ch^{-1/2}\|R\underline{V}^h\|_{0,\Gamma} \quad \text{for any } \underline{V}^h \in V_r^h. \tag{4.5}$$

Our error estimate is contained in the following theorem.

Theorem 2. *Suppose V_r^h approximates optimally with respect to r and satisfies* (4.5). *Let $[\underline{u}, p]^t$ be the solution of* (1.1). *Let $\omega = \operatorname{curl} \underline{u}$, $\underline{U} = [\underline{u}, \underline{\omega}, p]^t$, and $\underline{U}^h \in V_r^h$ be the solution of* (2.1). *Then*

$$\|\underline{U} - \underline{U}^h\|_1 \leq Ch^r\|\underline{U}\|_{r+1}.$$

Proof. Using (3.10) with $l = 1$, (4.5), and (4.1), we have for any $\underline{V}^h \in V_r^h$

$$\begin{aligned} \|\underline{V}^h\|_1^2 &\leq C(\|L\underline{V}^h\|_0^2 + \|R\underline{V}^h\|_{1/2,\Gamma}^2) \\ &\leq C(\|L\underline{V}^h\|_0^2 + h^{-1}\|R\underline{V}^h\|_{0,\Gamma}^2) = C \cdot a(\underline{V}^h, \underline{V}^h). \end{aligned}$$

Applying this inequality to $\underline{U}^h - \underline{V}^h \in V_r^h$ and using (4.4), we get

$$\begin{aligned}\|\underline{U}^h - \underline{V}^h\|_1^2 &\le Ca(\underline{U}^h - \underline{V}^h, \underline{U}^h - \underline{V}^h)\\ &= C(a(\underline{U}^h - \underline{U}, \underline{U}^h - \underline{V}^h) + a(\underline{U} - \underline{V}^h, \underline{U}^h - \underline{V}^h))\\ &= Ca(\underline{U} - \underline{V}^h, \underline{U}^h - \underline{V}^h) \le C_1\|\underline{U} - \underline{V}^h\|_1 \cdot \|\underline{U}^h - \underline{V}^h\|_1.\end{aligned}$$

Hence, $\|\underline{U}^h - \underline{V}^h\|_1 \le C\|\underline{U} - \underline{V}^h\|_1$. Using the optimal approximation property of $\underline{V}_r^h$, we choose $\underline{V}^h$ so that $\|\underline{U} - \underline{V}^h\|_1 \le Ch^r\|\underline{U}\|_{r+1}$. Then $\|\underline{U}^h - \underline{V}^h\|_1 \le Ch^r\|\underline{U}\|_{r+1}$, and so

$$\|\underline{U} - \underline{U}^h\|_1 \le \|\underline{U} - \underline{V}^h\|_1 + \|\underline{U}^h - \underline{V}^h\|_1 \le Ch^r\|\underline{U}\|_{r+1},$$

which is the desired result. □

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